EN MASSE BEHAVIOUR OF GRANULAR MATERIALS IN SILO LOAD CALCULATIONS

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ABSTRACT

Calculations of silo structure design problems, such as the evaluation of the interaction between the loads imposed on thin walled storage structures by grains and deformation of the storage structure wall or the evaluation of loads imposed on confining structures by particulate media during flow, would be greatly enhanced by accurate description of the load-deformation behaviour of stored particulate media en-masse. Early models represent situations which are extremely idealized versions of real life counterparts. The solutions are those for various simple stress distributions of linearly elastic isotropic homogeneous media on the one hand, and certain results derived from the upper bound methods of ideally plastic analysis on the other hand. This paper discusses mechanical properties of granular materials and their impact on load distribution in silos.

Keywords: granular material; silo; pressure distribution; elastic, plastic, viscoelastic, elastoviscoplastic

INTRODUCTION

The solution to grain storage structure design problems, such as the evaluation of the interaction between the loads imposed on thin walled storage structures by grains and deformation of the storage structure wall or the evaluation of loads imposed on confining structures by particulate media during flow, would be enhanced by description of the load-deformation behaviour of biological particulate media en-masse. Manbeck and Nelson (1972). However, there is controversy as to whether such materials are elastic, viscoelastic, elastoplastic or elastoviscoplastic.

Granular materials, which include everything from coal to coco pops, soils and all cereal grains, are large conglomerations of discrete macroscopic particles that do not quite fit into any of the known phases of matter: solid, liquid, or gas (Heinrich et al., 1996). The changing personalities of granular materials can have devastating implications, for example the disturbance of the earth following an earthquake can be enough to trigger solid ground to turn to mush with catastrophic consequences. Tordesillas (2004) asserts that, "Even a fractional advance in our understanding of how granular media behave can have a profound impact on the economic and general well-being of nations worldwide." Yet, despite being second only to water on the scale of priorities of human activities and believed to account for ten per cent of all energy consumed on earth, the physics behind granular materials remain largely unknown. Scientists have generally turned to the continuum theory for predicting the behaviours of solids, liquids and granular media (Harris, 2009) – it looks at an object as a whole rather than the sum of its parts. The alternative of modeling every single grain, as in the discrete element method (DEM), is computationally intensive and extremely costly.

Knowledge of constitutive relations in granular agricultural materials is of utmost importance in transportation, processing and in the design of processing and storage structures such as silos and bins. Jansen’s theory, for example, is
commonly used in most international standards for silo design (Moya et al., 2002). This theory, as well as many others such as Airy’s theory or Reimbert’s theory considers some material properties such as the angle of internal friction, the grain wall friction coefficient and the specific weight. Hence, it is possible to find values for all these properties in literature to apply in design (Mohsenin, 1980). However, to accurately model silo loads, it is necessary to consider additional material properties not taken into account in the traditional methods (Moya et al., 2002; Vanel et al., 2000). Constitutive models continue to rely on the theory of elasticity to predict the load-deformation behaviour of granular materials (Harris, 2009).

In an attempt to understand this behaviour, previous studies of rheology and strength of agricultural materials have focused on loading specimens to failure at a constant deflection rate or applying impact loading (Mohsenin, 1980). It is however well known that loading below the yield stress in metals and other composite engineering materials can eventually result in failure (Stinchcomb, 1989). Mclaughlin and Pitt (1984) showed that apple tissue could fail under cyclic or static loadings of magnitudes insufficient to cause failure initially. The question that arises is whether these small, repeated loads (cyclic loading) applied to bulk shelled maize can cause eventual damage as occurs in metal fatigue.

### 1.1 Classification of Granular Materials

Based on ISO classification (ISO 3535, 1977), the Mechanical Handling Engineers' Association (MHEA), UK divided the granular material according to the dimension of particles ($D$) into ten categories; however Chattopadhyay et al. (1994) proposed the five following groups:

- dust $D \leq 0.42$ mm
- grain $D \leq 3.35$ mm
- lump $D \leq 40$ mm
- clump $D \leq 200$ mm
- block $D > 200$ mm

The classification of granular materials according to bulk density ($\rho$) is as follows:

- light $\rho < 600$ kg m$^{-3}$
- medium $600$ kg m$^{-3} < \rho \leq 1100$ kg m$^{-3}$
- heavy $1100$ kg m$^{-3} < \rho \leq 2000$ kg m$^{-3}$
- very heavy $\rho > 2000$ kg m$^{-3}$

The flowability defined as a motion of particles with reference to neighbouring particles or along surfaces is the next parameter describing granular materials (Peleg, 1985). It has a huge influence on processes occurring during storage and handling of materials in industry and agriculture. The conventional classifications of materials according to flowability are derived from the classification based on the flow function ($FF$) proposed by Jenike (1961). Chattopadhyay et al., (1994) extended Jenike’s (1961) classification adding two extreme classes:

- fluidlike flooding
- very free flowing $FF > 10$
- free flowing $10 > FF > 4$
- average flowing $4 > FF > 2$
- poor flowing $2 > FF$
- sluggish/interlocked

In the case of agricultural and food raw materials and products, apart from the classifications mentioned above, attention should be paid also to a number of additional features, such as:

- possibility of freezing,
- hygroscopicity,
- toxic properties,
- properties conducive to spontaneous combustion and
- explosive properties

### Stress-Strain Models

Studies of the responses of cereal grains to mechanical stresses have progressed from analyses based on simple Hookean models to viscoelastic models using generalized Kelvin and Maxwell models, used singly or jointly Herum et al., (1979). More recently elastoplastic models have been used Zhang et al., (1986). Upon repeated uniaxial loading, Shapolyanskaya (1952) found that
the load-deformation behavior of wheat kernels approached that of an elastic body. Thus, applying the Hertz theory of contact stresses, he evaluated a modulus of elasticity for wheat kernels. In his study of core samples of wheat kernels, Zoerb (1960) noted the same strain hardening tendencies but concluded that plastic rather than elastic behavior characterized the mechanical properties of wheat.


While some earlier researchers treated the materials as elastic materials are very complex and may be defined as being elastoviscoplastic, Li et al., (1989).

2.1 Elastoplastic Models

The elastoplastic theory developed by Lade (1977) for cohesionless sand and verified for wheat en masse by Zhang et al., (1986) has the following form;

\[ \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p + \varepsilon_{ij}^c \]  

… (2.1)

Where:

\( \varepsilon_{ij} \) is the total strain increment tensor, and superscripts e, p, and c, refer to the elastic, plastic and plastic collapse strain increment components respectively. The complete elastoplastic constitutive relationship includes an elastic component and two work hardening plastic components. Figure 2.1 show basic elastoplastic components as described by Lade’s model.

\( \varepsilon_{ij} \) is the volumetric strain increment

Lade and Nelson developed a procedure to construct an incremental elastoplastic model with multi-intersecting yield surfaces. By using this Lade-Nelson method Chi and Kushwaha (1990) derived the complete elastoplastic model as presented below;

2.1.1. Lade’s complete Elastoplastic Constitutive Equation

The complete elastoplastic stress-strain equation developed by Lade (1977) may be written in incremental form as;

\[ d[\sigma] = [E_{\text{ep}}]d[\varepsilon] \]  

… (2.2)

\[ E_{\text{ep}} = E \left[ \frac{1}{A} \left( \frac{\partial g_e}{\partial \sigma} \right) \right] + \left( \frac{\partial g_p}{\partial \sigma} \right) \cdot (bp)'[E] \]  

… (2.3)

The terms in equation (3.3) are defined below;

\[ A = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \]  

… (2.4)

\[ b_c = L_{22} \left( \frac{\partial f_c}{\partial \sigma} \right) - L_{12} \left( \frac{\partial f_p}{\partial \sigma} \right) \]  

… (2.5)

\[ b_p = L_{11} \left( \frac{\partial f_p}{\partial \sigma} \right) - L_{21} \left( \frac{\partial f_c}{\partial \sigma} \right) \]  

… (2.6)

\[ L_{11} = \left( \frac{\partial f_c}{\partial \sigma} \right)^t \cdot [E] \cdot \left( \frac{\partial g_c}{\partial \sigma} \right) + \left( \frac{\partial f_p}{\partial \sigma} \right)^t \cdot \left( \frac{\partial g_c}{\partial W_p} \right) \cdot \left( \frac{\partial g_c}{\partial \sigma} \right) \]  

… (2.7)

\[ L_{22} = \left( \frac{\partial f_p}{\partial \sigma} \right)^t \cdot [E] \cdot \left( \frac{\partial g_p}{\partial \sigma} \right) + \left( \frac{\partial f_p}{\partial W_p} \right) \cdot \left( \frac{\partial g_p}{\partial \sigma} \right) \]  

… (2.8)

Alternatively,

\[ L_{11} = \left( \frac{\partial f_c}{\partial \sigma} \right)^t \cdot [E] \cdot \left( \frac{\partial g_c}{\partial \sigma} \right) \]  

… (2.9)

\[ L_{22} = \left( \frac{\partial f_p}{\partial \sigma} \right)^t \cdot [E] \cdot \left( \frac{\partial g_p}{\partial \sigma} \right) \]  

… (2.10)

All factors have their usual meaning.
The elastoplastic constitutive matrix (2.2) can readily be determined if the stress state is known.

The Mohr-Coulomb failure criterion finds many applications in engineering practice (Ti, 2009; Dartevelle, 2003). Analysis of stress data from triaxial cell tests on granular materials often involves the Mohr Circle construction to graphically determine strength parameters (Stafford et al., 1986). The Mohr circle construction helps to determine the stresses on any given plane.

From a series of tests, the minor ($\sigma_3$) and major ($\sigma_1$) principal stresses may be derived. On a graph with shear stress along the y-axis and normal stress along the x-axis, a circle centered on $x = (\sigma_1 + \sigma_2)/2$ of radius $(\sigma_1 - \sigma_2)/2$ is drawn for each test. A smooth curve, tangent to these circles defines the equation for the failure characteristic of the material in question. The ideal Mohr-Coulomb failure plot is shown in Figure 2.2.

**Figure 2.1:** A schematic representation of elastic, plastic collapse and plastic expansive strain components determined in triaxial tests (Lade, 1977).
The basic equation if of the form,
\[ \tau = c + \sigma \tan \varphi \]  \hspace{1cm} \ldots (2.11)

Where:
- \(\tau\) is shear stress
- \(\sigma\) is normal (principal) stress
- \(c\) is cohesion
- \(\varphi\) is angle of internal friction, all measured at failure.

The alternative to the above method of plotting circles is to directly plot \((\sigma_1 - \sigma_2)/2\) against \((\sigma_1 + \sigma_2)/2\) for each test and draw a best-fit straight line through the points. From this so-called \(K_f\) line, \(c\) and \(\varphi\) may be derived (Stafford et al., 1986). For cohesionless granular materials such as shelled maize, the cohesion, \(c\), is usually assumed to be negligible (Zhang et al., 1994).

Horabik and Molenda (2014) presented an analysis based on the Discrete Element Method (DEM) proposed by Cundal (1979) based on elementary interactions between the grains. The method consists in a simplified solution of the equation of motion for each grain of the material.

The calculation procedure is based on the assumption that during a very short time step \(\Delta t\) acceleration and speed are constant, and the disturbance of motion of a single grain does not reach further than to the nearest neighbours. DEM provides new possibilities of deeper insight into the micro-scale behaviour of bulk solids, which are not available with traditional or even modern approach of continuum mechanics where gradients of displacement and stress are extremely high like during flow around inserts (Kobyła, Molenda, 2013).

The rapid development of computer calculation techniques permitted the realization of computer simulations of a variety of processes occurring in granular materials, such as: dynamic effects in silos, mixing, segregation, gravitational discharge from silos (Zhang et al., 1993; Masson, Martinez, 2000; Koubu et al., 2002; Parafiniuk et al., 2013; Kobyła and Molenda, 2014).

DEM simulations generally produce a huge scatter of inter-particle forces which after averaging provide useful information. An example of the horizontal forces acting on a vertical wall in quasi-static assemblies (6000 particles in two dimensions) is presented in Figure 2.3. Analysis of the distribution of horizontal forces averaged for 10 particle–wall contacts indicated a moderately smooth increase in the force with increase in particle bedding depth (Sylwia et al., 2008). The DEM values are considerably larger as compared to Janssen’s (1895) solution. Similarly, Balaevich et al., (2011) obtained good agreement of lateral pressure distribution vs material depth with experimental data which were significantly larger than Janssen’s (1895) solution and Eurocode 1 (2003) recommendations. González-Montellano et al. (2012) obtained the pressure distribution of particles similar to maize grains along the vertical direction of the wall reaching its maximum at the silo-hopper transition using DEM. Masson, Martinez (2000) reported on the impact of anisotropy of contact orientations on the pressure distribution.
Horabik and Moleda (2014) concluded that granular solids constitute a very wide group of materials of specific mechanical, physical, and chemical properties. In that group granular materials of biological origin represent a very important source of products of agriculture and food processing industry. Precisely determined parameters of mechanical properties of granular materials and properly chosen constitutive models are fundamental for proper design and control of storage, handling, and processing of those materials. There have still been a lot of particular and specific operations not fully understood or precisely described yet. A few examples of such processes discussed in the paper indicate that further development of tools for modelling the mechanics of granular solids is necessary. One of the most promising tools is the DEM. All these actions are necessary to ensure that the equipment used for storage and processing of granular materials meets two basic demands: predictable and safe operations and high quality of processed materials.

**Viscoelastic Models**

The linear and non-linear viscoelastic models have been predominantly used to model material behaviour (Gumbe, 1995). These models incorporate time in stress-strain equations. Such equations are specifically important in the design of storage structures such as silos and foundations both of which are exposed to rapidly declining or changing periodic ambient temperatures, moisture content, among others (Gumbe, 1995).

For linearly viscoelastic material, the following criteria hold (Findley et al., 1976).

1. For any step input in strain \( \varepsilon_0 \) the relation between the stress \( \sigma(t) \) and strain is;

\[
\frac{\sigma(t)}{\varepsilon_0} = E(t)
\]

\[\text{… (2.12)}\]

For a step input of stress, \( \sigma_0 \), the relation between strain, \( \varepsilon(t) \) and stress is;
\[ \frac{\varepsilon(t)}{\sigma_0} = J(t) \]  \quad \text{... (2.13)}

Where \( E(t) \) and \( J(t) \) are the stress relaxation modulus and creep compliance respectively.

2. The Boltzmann’s superposition principle holds, that is, the stress at any time \( t \) depends on the strain history of the material;

\[
\sigma(t) = \int_0^t E(t - \tau) \frac{\delta \varepsilon}{\delta \tau} + \sum_{i=1}^{n} \Delta \varepsilon_i E(t - t_i)
\]

\quad \text{... (2.14)}

Where:
- \( \varepsilon(t) \) is the applied strain
- \( \Delta \varepsilon_i \) is the jump in the applied strain occurring at time, \( t_i \)

Most viscoelastic models are built on the spring-dashpot arrangement as the basic building blocks. The two basic models are the Kelvin solid model and the Maxwell fluid model (Chung, 1988). Figure 2.4 a show the Kelvin model with the spring and dashpot arranged in parallel while Figure 2.4 b shows the Maxwell model where the spring and dashpot are arranged in series.

The two basic equations governing the stress-relaxation behaviour of the Kelvin and Maxwell models at constant strain are given in (2.15) and (2.16) respectively;

\[
E(t) = E \quad \text{... (2.15)}
\]

\[
E(t) = E e^{-t/\tau} \quad \text{... (2.16)}
\]

Where:
- \( E(t) \) are the relaxation modulus and \( E \) the spring constants

The Maxwell and Kelvin models have been noted to have several deficiencies in predicting the behaviour of viscoelastic materials (Chung, 1988; Gumbe, 1993). These deficiencies include,

- Neither the Maxwell nor Kelvin model represents the behaviour of most viscoelastic materials in actual systems.
- The Maxwell model shows no time-dependent recovery nor does it show the decreasing strain rate under constant stress, a characteristic of primary creep.
- The Kelvin model does not exhibit time-independent strain on loading, nor does it describe a permanent strain after unloading.
- Both models show a finite initial strain rate where the initial strain rate for many materials is very rapid.

Due to the above limitations of the simplified models, several other viscoelastic models have been adopted to help make more accurate predictions of viscoelastic behaviour of engineering materials (Chung, 1988; Aklonis and MacKnight, 1983; Gumbe, 1993). One such model, which has been found to be fairly accurate, is the generalized Maxwell-Weichert model (Aklonis and MacKnight, 1983) that consists of an arbitrary number of Maxwell elements connected in parallel as shown in Figure 2.5.

![Figure 2.4: Basic Viscoelastic models](image-url)
Figure 2.5: Generalised Maxwell-Weichert model (Aklonis and MacKnight, 1983)

The generalized stress-relaxation equation for the Maxwell-Weichert model is of the form:

$$E(t) = \sum_{i=1}^{z} E_i e^{-k_i t} \quad \ldots (2.17)$$

Where
- $E_i$ is the relaxation modulus (MPa)
- $t$ is the time of relaxation (minutes)
- $k_i$ is the exponent (1/minute)

For practical purposes however, equation (2.17) has to be presented to finite levels. Equation (2.18) therefore is the representation of (2.17) at two levels (Herum, 1979):

$$E(t) = E_1 e^{-k_1 t} + E_2 e^{-k_2 t} \quad \ldots (2.18)$$

The two equations, (2.17) and (2.18), describe the Maxwell-Weichert linear viscoelastic model as long as the constants $E_1$, $E_2$, $k_1$ and $k_2$ are determined.

Oranga (2005) conducted test to evaluate relaxation parameters which included the stress-relaxation moduli, $E_1$, $E_2$ and the stress-relaxation exponents’ $k_1$, $k_2$. Figures 2.6-2.8 show the stress relaxation curves for the three maize varieties, at an initial bulk density of 800 kg/m$^3$ (three replications) and the fitted curves according to the given regression equations indicated.

Figure 2.6: Stress-relaxation curve for variety V1
Figure 2.7: Stress-relaxation curve for variety V2

Figure 2.8: Stress-relaxation curve for variety V3
Figure 2.9: Coefficient of determination, $R^2$, for $E(t)$, variety V1

Figure 2.10: Coefficient of determination, $R^2$, for $E(t)$, variety V2

Figure 2.11: Coefficient of determination, $R^2$, for $E(t)$, variety V3
The standard error on the residuals for each of the varieties V1, V2 and V3 were found to be 1.17, 0.95 and 1.82 MPa respectively. Similarly, the coefficient of determination between the measured values and the fitted values were obtained as 0.87, 0.84 and 0.90 for V1, V2 and V3 respectively as shown in Figures 2.9-2.11. These were indicators that the measured data and the regressed values fitted quite closely statistically.

The flexibility of the Maxwell-Weichert model in reproducing the viscoelastic behaviour of shelled maize en masse was further demonstrated by plotting the above results on a log-log scale as shown in Figure 2.12.

Figure 2.12, taken from one of the replications for V1, shows the application of the generalized form of Maxwell-Weichert model described in equation 2.17 with an almost infinite number of parameters (observed on the step-decline of the curve). Such behaviour, which was observed for all the varieties, has been reported for polymer viscoelastic materials by Aklonis and MacKnight (1983).

**Elastoviscoplastic Models**

Li et al., (1989) presented a study based on the constitutive equation based on the elastic-viscoplastic (EVP) theory for cohesionless sands (Youngs, 1982) to model the stress-strain behaviour of wheat en masse under monotonic and cyclic loading conditions. This EVP model considers:

I. The effect of confining pressure or hydrostatic pressure on the deformation and strength characteristic of the particulates;

II. The development of large inelastic shear deformations during the loading process; and

III. The coupling between inelastic shear and volumetric deformation with changing stress state.

The three-dimensional constitutive formulation incorporates the effect of the evolution of intergranular structure on both stress-strain response and inelastic volume change induced by shear deformation. Stress-strain behaviour predicted by the Youngs' EVP model is based on an associated plasticity formulation for deviatoric strains and a separate constitutive equation for coupled volumetric strains. The parameters of the constitutive equations are assumed to be a function of void ratio, or bulk density, and the mean stress both of which reflect the effects of loading history. The parameter calculation technique used in the study was similar to that described by Youngs (1982).

### 2.2.1. One Dimensional Constitutive Equation

The key features of the elastic-viscoplastic (EVP) constitutive equation developed by Youngs (1982) are explained through the one-dimensional formulation.

**Figure 2.12**: Behaviour of Maxwell-Weichert model in stress-relaxation for V1
For further in-depth understanding of the stress transfer and displacement mechanisms involved in elastic viscoplastic constitutive models the reader may consult these references (Chen and Baladi, 1985; Feda, 1982; and Youngs, 1982). The one-dimensional, stress-strain model is used as the stepping stone for presenting the three-dimensional model.

In the one-dimensional model, two strain rate equations (based on particulate theory) one for shear strain and one for volumetric strain, are used to characterize the grain en masse behaviour. The shear strain rate \( \dot{\gamma} \) equation consists of two components: 1) elastic shear strain rate \( \dot{\gamma}_e \) and 2) viscoplastic shear strain rate \( \dot{\gamma}_p \). The elastic shear strain rate is the recoverable component of strain; whereas, the viscoplastic component is the irrecoverable (permanent) component of strain. The following shear strain rate equation was derived by Youngs (1982):

\[
\dot{\gamma}_e = \dot{\gamma}_e + \dot{\gamma}_p = \frac{\tau}{G} + |\dot{\gamma}| \left( \frac{\tau - \tau^r}{|\tau| - \tau^r} \right)^m \frac{\tau}{|\tau|} \tag{2.19}
\]

Where:

- G is shear elastic modulus,
- m is plastic hardening exponent,
- \( \dot{\tau} \) is shear stress rate,
- \( \tau \) is shear stress,
- \( \dot{\gamma}_e \) is elastic shear strain rate,
- \( \dot{\gamma}_p \) is viscoplastic shear strain rate.

In equation 2.19 the ratio \( \dot{\tau}/|\tau| \) is the unit vector that provides the stress direction of plastic flow, whereas, \( |\dot{\gamma}| \) with the dimension of reciprocal time implicitly introduces the time factor in the formulation. In equation 2.19, the yield stress \( \tau^y \) identifies the onset of plastic flow, and \( \tau^y \) is the state point on the stress surface where the stress direction rate \( (\dot{\tau}) \) was last reversed, i.e., changed from increasing to decreasing rate or vice-versa. The stress reversal point becomes a memory parameter, which is used to define the relative distance to the yield stress in the direction of stress rate (Ozdemir, 1976). The yield surface expands or contracts depending on the state of stress (Chen and Baladi, 1985). The volumetric strain rate \( \dot{e}_v \) equation consists of two components: 1) the expansion strain rate \( \dot{e}_v^x \) and 2) the collapse strain rate \( \dot{e}_v^c \). The expansion volumetric strain rate is due to the packing density change; whereas, the collapse volumetric strain rate is due to the grain reorientation. The volumetric strain rate equation is:

\[
\dot{e}_v = \dot{e}_v^x + \dot{e}_v^c = |\dot{\gamma}| [A(e - e_{cs})\lambda + B(1 - \lambda)] \tag{2.20}
\]

Where: A is volumetric expansion parameter, B is volumetric collapse parameter, \( e \) is current void ratio, \( e_{cs} \) is critical state void ratio, \( \lambda \) is relative structure change parameter \( (0 \leq \lambda \leq 1) \).

In equation 2.19, the yield stress \( \tau^y \) in the direction of loading was hypothesized by Youngs (1982), based on experimental evidence, to be a function of the relative structure change parameter:

\[
\tau^y = \tau^{f^-} + (\tau^{f^+} - \tau^{f^-}) f(\lambda) \\
= (\tau^{f^+} - \tau^{f^-}) [1 - \exp(-5\lambda)] \tag{2.21}
\]

Where:

- \( \tau^{f^+} \) is failure shear in the direction of shear (e.g., increasing or decreasing rate)
- \( \tau^{f^-} \) is failure shear stress in the opposite direction of shear (e.g., decreasing or increasing rate, respectively).

In equation 2.21, \( f(\lambda) \) is an empirical function derived from experimental data. The function \( f(\lambda) \) for wheat was arrived at \([1 - \exp(-5\lambda)]\) using the triaxial test data and Youngs (1982) linearity argument. For sands, exponents greater than 3 (compared to 5 for wheat) have been reported (Youngs, 1982). The relative structure change parameter \( \lambda \) is expressed as:
\[
\lambda = 1 - \left[ \frac{\left( \gamma / |\gamma| \right) - e^s}{2} \right] -1 \leq e^s \leq 1 
\]

... (2.22)

Where \( e^s \) is the variable used to measure the grain en masse structure.

The magnitude of the relative structure change parameter (\( \lambda \)) indicates the location of the failure stress state of the particulate material as the stress-strain path follows the limiting hardening curve, i.e., \( \tau = \tau^s \) at all times. The value of \( \lambda \) varies from zero at the stress reversal point at the failure stress in one direction, to a value of unity as the failure stress, and is reached in the direction of shear. The magnitude of \( e^s \) measures the development of relative structure change and its sign determines the direction of the relative structure change. The absolute value of \( e^s = 1 \) implies that the particle rearrangement has been fully accomplished. This results in volumetric changes in collapse mode. Any further volumetric changes are due to the packing density adjustments. The rate of \( e^s \) is defined by:

\[
\dot{e}^s = C \dot{\gamma} (1 - \lambda^2) \quad ... (2.23)
\]

Where;

\( \dot{e}^s \) is the rate of relative structure change,

\( C \) is structure modulus which controls the hardening rate,

\( l \) is exponent for the measure of linearity of relative structure change.

A useful interpretation of equation 2.23 is that initially \( \dot{e}^s \) is linearly proportional to \( \dot{\gamma} \) since (\( \gamma \approx 1 \)) no further changes in \( e^s \) occur during continued shear in that direction.

### 2.2.2. Three-Dimensional Constitutive Equation

Two assumptions, as postulated by Youngs (1982), are herein retained to expand the one-dimensional model described in the preceding section to a three-dimensional formulation. The assumptions are: 1) the granular material is isotropic, and 2) separate relationships can be developed for the response to deviatoric and hydrostatic (isotropic) loading.

The first assumption of isotropy simplifies the constitutive equations in that only two elastic material parameters (e.g., shear modulus and Poisson's ratio) are needed to completely describe the stress-strain relationship. In addition, invariants can also be effectively used to establish the state of stress. The evidence of structural anisotropy in sands has been demonstrated by Oda et al., (1978). Tests by Lade (1972) and Oda et al., (1972) with sand particles of various shapes showed that anisotropic behaviour decreases as the particle sphericity of the wheat particles tested in the present study was 55%. Hence, the load response of wheat en masse is expected to be anisotropic. The constitutive equation can be modified using the full elasticity tensor and introducing weighting factors in calculating the stress invariants as suggested by Sandler and DiMaggio (1973). The fully anisotropic constitutive equation requires 21 material constants for elastic component alone (Green-type linearly elastic material), which are extremely difficult if not impossible to determine. Therefore, as a first approximation and to keep the total number of material parameters at a reasonable number, the material response is assumed to be isotropic.

The deviatoric strain rate equation in three-dimensions similar in form to the one-dimension equation 2.19 is:

\[
\dot{\varepsilon}_{ij} = \frac{S_{ij}}{2\alpha} + \|\varepsilon_{ij}\| \left( \frac{|S_{ij} - S_{ij}^r|}{|S_{ij} - S_{ij}^r|} \right)^m n_{ij} 
\]

... (2.24)

Where;

\( \dot{\varepsilon}_{ij} \) Is rate of deviatoric strain, \( \|\varepsilon_{ij}\| \) is norm of deviatoric strain rate, which is also the length of the tensor \( \varepsilon_{ij} \) projected on the octahedral plane and is defined by eq.2.24, \( m \) is shear yield exponent, \( n_{ij} \) is direction tensor, similar to the ratio \( \frac{\tau^s}{|\tau^s|} \) in equation 2.19, \( S_{ij} \) is deviatoric stress, \( S_{ij}^r \) is rate of deviatoric stress, \( S_{ij}^r \) is deviatoric stress reversal point and \( S_{ij}^2 \) is yield deviatoric stress,
\[
\|\dot{\epsilon}_{ij}\| = (\dot{\epsilon}_{ij}\dot{\epsilon}_{ij})^{1/2} = (2\|\dot{\epsilon}_{ij}\|)^{1/2}
\]  
\]  
\[\|\dot{\epsilon}_{ij}\| \text{ is second invariant of deviatoric strain}\]

The double subscript notation has the usual meaning, i.e. the first index denotes the plane normal direction and the second index gives the direction of the tensorial quantity.

Both \(i\) and \(j\) subscripts can take on values 1, 2, and 3 which correspond to \(x\), \(y\), and \(z\) axis directions, respectively. In equation 2.25, repeated indices appearing in the subscripts imply summation, e.g. \(e_{ii} = e_{11} + e_{22} + e_{33}\).

Since any hydrostatic (isotropic) stress change from a stress state containing a deviatoric component results in a movement relative to the failure surface (Lade and Duncan, 1971), normalized deviatoric stress \(\bar{S}_{ij}\), normalized deviatoric strain rate \(\dot{\bar{S}}_{ij}\) and normalized shear modulus \(\bar{G}\) are defined as follows (the overbar symbol, \(\bar{\cdot}\), is used to denote normalized quantities):

\[
\bar{S}_{ij} = \frac{S_{ij}}{(\sigma_m/P_a)^a}
\]  
\[\text{... (2.26)}\]

\[
\dot{\bar{S}}_{ij} = \frac{S_{ij}}{(\sigma_m/P_a)^a} \dot{\epsilon}_{ij}
\]  
\[\text{... (2.27)}\]

\[
\bar{G} = \frac{G}{(\sigma_m/P_a)^{2a}}
\]  
\[\text{... (2.28)}\]

Where:

\(P_a\) is atmospheric pressure, \(\sigma_m\) is mean stress which is defined as \(\sigma_m = \frac{1}{3} \sigma_{kk}\)

\(\alpha\) is failure surface curvature parameter

The deviatoric strain rate given by equation 2.24 then becomes:

\[
\dot{\epsilon}_{ij} = \frac{\dot{S}_{ij}}{2\bar{G}} - \alpha \frac{\dot{S}_{ij}}{2\sigma_m} \frac{\sigma_m}{\sigma} + \|\dot{\epsilon}_{ij}\| \left(\frac{\|\dot{S}_{ij}-\dot{S}_{ij}\|}{\|\dot{S}_{ij}-\dot{S}^-_{ij}\|}\right)^m n_{ij}
\]  
\[\text{... (2.29)}\]

In equation 2.29, normalized yield \((\dot{S}^Y_{ij})\) and reversal \((\dot{S}^-_{ij})\) stresses appear because of the symmetry of response which results from the use of normal stresses. The directional tensor \(n_{ij}\) defines the direction of inelastic shear on the octahedral plane, and \(n_{ij}\) is the normal to the failure surface at a point defined by the direction of shear.

Using similar reasoning, a hydrostatic strain rate equation was derived by Youngs (1982):

\[
\dot{\varepsilon}_{kk} = \frac{\dot{\sigma}_m}{K^e} + |\dot{\varepsilon}_{kk}| \bar{K}^p \left(\frac{\sigma_m - \sigma_m^{max}}{\sigma_m^{max} - \sigma_m}\right)^h
\]  
\[\text{... (2.30)}\]

Where;

\(K^e\) is elastic bulk modulus, \(\bar{K}^p\) is plastic bulk modulus, \(h\) is plastic bulk exponent, \(\dot{\varepsilon}_{kk}\) is hydrostatic volumetric strain rate, \(\dot{\sigma}_m\) is mean stress rate, \(\sigma_m^{max}\) is past maximum mean stress rate.

The volumetric strains produced by three-dimensional shear straining are modelled the same way as the one-dimensional shear given by equation 2.20. The rate equation for volumetric strain becomes:

\[
\dot{\varepsilon}_{kk} = \|\dot{\epsilon}_{ij}\| \left[A(e - e_{cs})\lambda + B(1 - \lambda)\right]
\]  
\[\text{... (2.31)}\]

The total volumetric strain rate is the sum of hydrostatic (isotropic) volumetric strain rate (eq 2.30) and shear volumetric strain rate (eq. 2.31) In equation 2.24, the yield deviatoric stress \(S^Y_{ij}\) is defined by equations 2.32 and 2.34 which is similar to the one-dimensional equation 2.21. The form of the empirical function \(f(\lambda)\) for the three-dimensional model suggested by experiments was \(\lambda^N\) (Youngs, 1982).

\[
S^Y_{ij} = S^+_{ij} + \left(\frac{S^Y_{ij} - S^-_{ij}}{\sigma_m}\right)^{\lambda^N}
\]  
\[\text{... (2.32)}\]

\[
\bar{N} = N_1 + N_2 \frac{\|\dot{S}^+_{ij}\|}{\|\dot{S}^-_{ij}\|}
\]  
\[\text{... (2.33)}\]

Where;

\(S^+_{ij}\) is failure deviatoric stress in loading direction, \(S^-_{ij}\) is failure deviatoric stress in
opposite loading direction, $N_1$ and $N_2$ are hardening curve exponents.

The structure change $\lambda$ in the three-dimensional space is defined as:

$$\lambda = 1 - \frac{\|\dot{e}_{ij}\|}{2} \quad \ldots (2.34)$$

The structure change tensor $\lambda_{ij}$ as in the one-dimensional case is defined in terms of the direction of shear strain. The tensor $\dot{e}_{ij}^s$ defines the current structure and $\dot{e}_{ij}$ the current strain rate. The continued shear in the direction of $\dot{e}_{ij}$ produces new structure in the direction $\dot{e}_{ij} / \|\dot{e}_{ij}\|$:

$$\lambda_{ij} = \frac{\dot{e}_{ij}}{\|\dot{e}_{ij}\|} - \dot{e}_{ij}^s \quad \ldots (2.35)$$

Where:

$\dot{e}_{ij}^s$ Is the relative structure change

The rate equation for $\dot{e}_{ij}^s$ in three dimensions which is similar to the one dimension $\dot{e}^s$ equation 2.23 is given by:

$$\dot{e}_{ij}^s = C\|\dot{e}_{ij}\|(1 - \lambda^i) \frac{\dot{e}_{ij}}{\|\dot{e}_{ij}\|} \quad \ldots (2.36)$$

Where $C$ is the structure modulus

The structure modulus rate equation using the arguments similar to the one-dimensional model is given by:

$$\dot{C} = C\|\dot{e}_{ij}\|[(1 - \lambda)D_2 - \lambda D_3]$$

$$\ldots (2.37)$$

Where $D_2$ and $D_3$ are relative structure change parameters.

Lee and Albaisa’s (1974) experimental results showed that shear straining causes a reduction in the hydrostatic yield stress ($\sigma_m^{\text{max}}$). They also observed that volumetric strains for reconsolidated sand samples were the same under small shear strains. However, the volumetric strains exceeded the initial consolidation under large shear strains (Ishihara and Okada, 1978). These results imply that reorientation of granular material structure during shear erases the memory of previous overconsolidation. To account for this effect the following equation was proposed by Youngs (1982):

$$\sigma_m^{\text{max}} = -D_1P_o(OCR - 1)\|\dot{e}_{ij}\| \quad \ldots (2.38)$$

Where; $\sigma_m^{\text{max}}$ is maximum mean stress rate, $D_1$ is plastic bulk modulus constant, OCR isoverconsolidation ration which is defined as:

$$OCR = \frac{\sigma_m^{\text{max}}}{\sigma_m} \quad \ldots (2.39)$$

The three-dimensional Youngs' EVP model for grain en loading is described by the deviatoric and hydrostatic strain rate equations 2.29 and 2.30, respectively, the volumetric strain rate (due to shear) equation 2.31, the structure change equation 2.35, and two structure change rate evolution equations 2.36 and 2.37.

Li et al., (1990) made the following conclusions. Fifteen material parameters and their relationships for wheat en masse in Youngs' elastic-viscoplastic (EVP) model were determined from monotonic and cyclic triaxial test results. The single set of parameter values was used in the constitutive equations to successfully predict the cyclic load response in triaxial tests. The chi-square ($\chi^2$) statistical test was used to evaluate the EVP model at the 0.05 level of significance. The following conclusions were drawn from this study:

1. Monotonic and cyclic triaxial tests can be used to determine all parameters and their relationships in the Youngs' EVP model.
2. The $\chi^2$ test (at the 0.05 level of significance) established the goodness of Youngs' EVP model for predicting monotonic and cyclic stress (deviatoric and isotropic-strain response of wheat en masse when data for all cycles were pooled in one set.
3. The $\chi^2$ test (at the 0.05 level of significance) established the goodness of Youngs' EVP model for predicting monotonic volumetric strain response for all tests and cyclic volumetric strain response for the high-density test samples (849 kg/m$^3$) when data for all cycles were pooled in one set.
4. At the 0.05 level of significance, the \( \chi^2 \) test demonstrated that the deviatoric and isotropic responses predicted by the EVP model and the triaxial data were not statistically different for the first two cycles in all tests.

5. At the 0.05 level of significance, the \( \chi^2 \) test demonstrated that the EVP model predicted and the triaxial volumetric strain values were not statistically different for the first two cycles of loading at initial bulk density of 849 kg/m\(^3\).

**Table 2.1** Axial stress difference and isotropic stress \( \chi^2 \) values for wheat en masse*

<table>
<thead>
<tr>
<th>Confining pressure (kPa)</th>
<th>Monotonic loading 801 kg/m(^3)</th>
<th>Monotonic loading 840 kg/m(^3)</th>
<th>Cyclic loading 801 kg/m(^3)</th>
<th>Cyclic loading 840 kg/m(^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.7</td>
<td>21.65</td>
<td>20.15</td>
<td>25.92</td>
<td>25.64</td>
</tr>
<tr>
<td>34.5</td>
<td>20.74</td>
<td>19.97</td>
<td>25.07</td>
<td>24.17</td>
</tr>
<tr>
<td>48.3</td>
<td>20.91</td>
<td>18.23</td>
<td>24.81</td>
<td>23.29</td>
</tr>
<tr>
<td>62.1</td>
<td>19.26</td>
<td>19.57</td>
<td>24.40</td>
<td>24.60</td>
</tr>
<tr>
<td>Isotropic</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0–82.2</td>
<td>25.86</td>
<td>25.47</td>
<td>30.91</td>
<td>31.14</td>
</tr>
</tbody>
</table>

* At 0.05 level of significance and 50 degrees of freedom, \( \chi^2 \) value is 34.8

**Table 2.2** Volumetric strain \( \chi^2 \) values for wheat en masse for cycles*

<table>
<thead>
<tr>
<th>Confining pressure (kPa)</th>
<th>Monotonic loading 801 kg/m(^3)</th>
<th>Monotonic loading 840 kg/m(^3)</th>
<th>Cyclic loading 801 kg/m(^3)</th>
<th>Cyclic loading 840 kg/m(^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.7</td>
<td>34.09</td>
<td>33.05</td>
<td>37.01</td>
<td>34.49</td>
</tr>
<tr>
<td>34.5</td>
<td>32.71</td>
<td>34.28</td>
<td>38.24</td>
<td>33.50</td>
</tr>
<tr>
<td>48.3</td>
<td>33.23</td>
<td>31.05</td>
<td>34.82</td>
<td>34.79</td>
</tr>
<tr>
<td>62.1</td>
<td>31.78</td>
<td>31.22</td>
<td>38.76</td>
<td>33.26</td>
</tr>
</tbody>
</table>

* At 0.05 level of significance and 50 degrees of freedom, \( \chi^2 \) value is 34.8

**Table 2.3** Cycle by cycle \( \chi^2 \) values for isotropic stress for wheat en masse

<table>
<thead>
<tr>
<th>Cycle</th>
<th>Isotropic pressure (kPa)</th>
<th>Monotonic loading 801 kg/m(^3)</th>
<th>Monotonic loading 849 kg/m(^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0–20.7</td>
<td>27.6</td>
<td>29.3</td>
</tr>
<tr>
<td>2</td>
<td>0–41.4</td>
<td>27.1</td>
<td>28.5</td>
</tr>
<tr>
<td>3</td>
<td>0–62.1</td>
<td>35.0</td>
<td>36.7</td>
</tr>
<tr>
<td>4</td>
<td>0–82.8</td>
<td>36.2</td>
<td>38.2</td>
</tr>
</tbody>
</table>

* At 0.05 level of significance and 50 degrees of freedom, \( \chi^2 \) value is 34.8
CONCLUSIONS

The mechanical behaviour of materials stored in silos and bunkers influences the flowability of the material and the forces that the material applies to the silo walls and bottom. The major problem in structural design of silos is to predict, with reasonable accuracy, the loads that these structures will be required to withstand during their service life.

This paper has reviewed elastoplastic, viscoelastic and elastoviscoelastic constitutive models for granular materials. Precise constitutive models and properly evaluated mechanical properties of granular materials are fundamental for proper design and control of storage, handling, and processing of those materials. More work remains in this crucial study area.

REFERENCES


ISO 3535. 1977. *Continuous mechanical handling equipment – Classification and symbolization of bulk materials.* International Agrophysics, 14,385-392


